K-theory of (Finite) Fields

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Abstract

The goal of this seminar is to study Quillen's computation of the algebraic K-theory of finite fields. This makes use of various topological and algebraic tools from homological algebra, simplicial homotopy theory, topological K-theory, and group cohomology. These tools will be reviewed in as much detail as possible and depending on the audience's prior knowledge.
To give a talk or for general information, you can write me an email at: giacomo.bertizzolo "at" mathematik.uni-regensburg.de .

What we are going to study

K-theory of finite fields

Let R be a ring with identity. We define the general linear group of R to be the filtered colimit

$$GL(R) := \mathrm{colim}_n GL_n(R) \in \mathrm{Grp},$$

and $E(R)$ to be the subgroup of $GL(R)$ generated by elementary matrices (recall that an elementary matrix is a matrix which differs from the identity matrix by one single entry). The classifying space $BGL(R)$ is path-connected, and $E(R)$ is the largest perfect normal subgroup of $\pi_1(BGL(R)) = GL(R)$, so there exists an acyclic map

$$\eta : BGL(R) \to BGL(R)^+$$

unique up to homotopy equivalence such that $\ker \pi_1(f) = E(R)$. The map η has the following universal property: *for each space X such that the π_1 of each connected component contains no non-trivial perfect groups, we have*

$$\eta^* : [BGL(R)^+, X] \xrightarrow{\simeq} [BGL(R), X].$$

For $i \geqslant 1$, the i-th K-group of R is defined as

$$K_i(R) = \pi_i(BGL(R)^+).$$

The first goal is to prove the following theorem:

Theorem 1 (**Quillen, [Qui72b]**)
Let $\mathbf{F}_q$ denote the field with $q = p^d$ elements for a prime p. Then

$$\begin{aligned} &K_0(\mathbf{F}_q) = \mathbf{Z}, \\ &K_{2n-1}(\mathbf{F}_q) = \mathbf{Z}/(q^n - 1) \quad \forall n > 1, \\ &K_{2n}(\mathbf{F}_q) = 0 \quad \forall n > 0. \end{aligned}$$

Theorem 1 can be proved in many different ways. The original one [Qui72b] is based the following:

Theorem 2 (**[Qui72b]**)
There is a homotopy equivalence $\theta : BGL(\mathbf{F}_q)^+ \to F\psi^q$, where $F\psi^q$ is the space of homotopy-theoretical fixed points of the Adams operation ψ^q.

Given Theorem 2, for $i > 0$ we know

$$K_i(R) = \pi_i(BGLR^+) \simeq \pi_i(F\psi_q),$$

and we can compute the homotopy groups of $F\psi^q$ using Bott periodicity.
The primary references we are going to use to prove Theorem 1 is Quillen's original paper [Qui72b] and the expository notes [Mit].

Idea of proof.

Let k denote the finite field with $q = p^d$ elements, and let ℓ denote a prime different from p. The first task is to produceθ. Using the Brauer theory of modular characters, we can lift the standard representations of $GL_n(k)$ to a virtual complex representation; this corresponds to maps $BGL_n(k) \to BU$, where BU is the classifying space of $U := \mathrm{colim}_n U(n)$. The map $BGL_n(k) \to BU$ is unique up to homotopy.
The Adams operation, denoted $\psi^q : K(\bullet) \to K(\bullet)$ for $q \in \mathbf{Z}_{\geqslant 1}$, is a cohomology operation in topological K-theory, characterized by three properties:

- $\psi^q([L]) = [L^{\otimes q}] \in K(X)$ if $[L]$ the K-theory class of a line bundle over X for any X topological space.
- $\psi^q : K(X) \to K(X)$ is ring morphisms for any X topological space.
- ψ^q is a natural transformation of functors.

The virtual complex representation is fixed under the Adams operation ψ^q, and hence lifts to a map $BGL_n(k) \to F\psi^q$. As n varies, these maps will also fit together to give us the map $\theta : BGL(k) \to F\psi^q$. This produces by acyclicity the desired map $\theta : BGL(k)^* \to F\psi^q$.
$BGL(\mathbf{F}_q)^+$ and $F\psi^q$ are homotopy associative and commutative H-spaces, and so, in particular, simple spaces; therefore, by Whitehead's theorem, we reduce to proving that θ induces an isomorphism on integral homology. By standard homological algebra, it is then enough to show that θ induces an isomorphism on homology with coefficients in $\mathbf{Q}$, $\mathbf{Z}/p$, and $\mathbf{Z}/\ell$ (recall that $\ell \neq p$). We have the following claims:

(a) $\tilde{H}_*(F\psi^q; \mathbf{Q}) = 0 = \tilde{H}_*(F\psi^q; \mathbf{Z}/p)$.

(b) $\tilde{H}_*(BGL(\mathbf{F}_q)^+; \mathbf{Q}) = 0 = \tilde{H}_*(BGL(\mathbf{F}_q)^+; \mathbf{Z}/p)$.

(c) θ induces an isomorphism on homology with coefficients in $\mathbf{Z}/\ell$.

Claim (a) follows from Serre's "mod $\mathcal{C}$" theory. The first equality of claim (b) can be proved using "transfer" in group cohomology; the second equality requires more work and involves both techniques from group cohomology and the use of the Hochschild-Serre spectral sequence. The hardest part lies in claim (c) and requires the explicit computation of the cohomology groups mod ℓ for both spaces.

K-theory of algebraically closed fields

Let k be an algebraically closed field. Lichtenbaum conjectured (see [Ger73], see [Qui74]) and Suslin proved (see [Sus83], [Sus84]) that $K_i(k)$ is divisible, its torsion being zero when i is even, and equal to $W(n)$ when $i = 2n-1$, where W_n is the nth Tate twist of the group W of roots of unity in k^*. As noted by Suslin (see [Sus83]) this conjecture can be reduced to a conjecture concerning the structure of the cohomology ring $H^*(GL(k), \mathbf{Z}/\ell)$ with ℓ prime different from $char(k)$ and also to the conjecture concerning the structure of the K-theory ring $K_*(k, \mathbf{Z}/\ell)$.

- The (Lichtenbaum's) conjecture is true if k is the algebraic closure of a finite field by the work of Quillen [Qui72b].

Theorem 3 (**Quillen, [Qui72b]**)
Let $\mathbf{F}$ denote an algebraic closure of $\mathbf{F}_p$. Then for $i \geqslant 1$

$$K_i(\mathbf{F}) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ \bigoplus_{l \neq p} \mathbf{Q}_\ell/\mathbf{Z}_\ell & \text{if } i \text{ is odd.} \end{cases},$$

where $\mathbf{Q}_\ell$ and $\mathbf{Z}_\ell$ are the ℓ-adic completion of $\mathbf{Q}$ and $\mathbf{Z}$, respectively.
The automorphism of $K_{2i-1}(\mathbf{F})$ induced by the Frobenius map $x \mapsto x^p$ is the

multiplication by p^i. If $\mathbf{E}$ is a subfield of $\mathbf{F}$, then

$$K_*(\mathbf{E}) \xrightarrow{\simeq} K_*(\mathbf{F})^{Gal(F/E)}.$$

- In [Sus83] is shown that if k'/k is an extension of algebraically closed fields, then

$$K_*(k, \mathbf{Z}/n) \to K_*(k', \mathbf{Z}/n)$$

  are isomorphisms.

In [Sus84], it is shown that the case of one field of characteristic zero implies the conjecture for all fields of characteristic 0. The computation of the K-theory for $\mathbf{C}$ concludes the proof.

Theorem 4 (**Suslin, [Sus84]**)
Modulo uniquely divisible groups, we have

$$K_i(\mathbf{C}) \otimes \mathbf{Q} = \begin{cases} 0 & \text{if } i \text{ is even,} \\ \mathbf{Q}/\mathbf{Z} & \text{if } i \text{ is odd.} \end{cases}$$

More precisely, modulo uniquely divisible groups, the K-theory of the fields $\mathbf{R}$ and $\mathbf{C}$ are as displayed by the following table

i mod 8	0	1	2	3	4	5	6	7
$K_i(\mathbf{R})$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Q}/\mathbf{Z}$	0	0	0	$\mathbf{Q}/\mathbf{Z}$
$\downarrow$	0	incl.	0	mult. by 2	0	0	0	iso
$K_i(\mathbf{C})$	0	$\mathbf{Q}/\mathbf{Z}$	0	$\mathbf{Q}/\mathbf{Z}$	0	$\mathbf{Q}/\mathbf{Z}$	0	$\mathbf{Q}/\mathbf{Z}$

In [Sus84] there is also a proof of Lichtenbaum's conjecture for the positive characteristic case that does not use the computations of Quillen.

Further Topic: Rational K-theory of number fields

To conclude our seminar, we will look at results on number fields.

Theorem 5 (**Quillen, [Qui72a]**)
Let $\mathcal{O}_\mathbf{F}$ be the ring of integers of a number field $\mathbf{F}$. Then, the groups $K_i(\mathcal{O}_\mathbf{F})$ are finitely generated.

Theorem 6 (**Borel, [Bor74]**)
Let $\mathcal{O}_\mathbf{F}$ be the ring of integers of a number field $\mathbf{F}$. Let r and s be the numbers of real

and complex embeddings of F, respectively. Then,

$$K_i(\mathcal{O}_{\mathbf{F}}) \otimes \mathbf{Q} = \begin{cases} \mathbf{Q} & \text{if } i = 0, \\ 0 & \text{if } i = 1 \text{ or } i > 0 \text{ even}, \\ \mathbf{Q}^{r+s} & \text{if } i \equiv 1 \bmod 4 \text{ and } i > 1, \\ \mathbf{Q}^{r+s} & \text{if } i \equiv 3 \bmod 4. \end{cases}$$

Moreover,

$$K_0(\mathbf{F}) = \mathbf{Z},$$
$$K_1(\mathbf{F}) = \mathbf{F}^\times,$$
$$K_i(\mathbf{F}) \otimes_{\mathbf{Z}} \mathbf{Q} = K_i(\mathcal{O}) \otimes_{\mathbf{Z}} \mathbf{Q} \quad \forall i \geqslant 2.$$

Notice however that the groups $K_i(\mathbf{F})$ are not necessarily finitely generated: while $K_i(\mathbf{F}) \otimes_{\mathbf{Z}} \mathbf{Q} = K_i(\mathcal{O}) \otimes_{\mathbf{Z}} \mathbf{Q}$ there may be infinite torsion in $K_n(\mathbf{F})$. Clearly,

$$K_1(\mathbf{Q}) = \mathbf{Q}^\times,$$

is not finitely generated, but also

$$K_2(\mathbf{Q}) = \mathbf{Z}/2 \oplus \bigoplus_{p \text{ prime } \geqslant 3} (\mathbf{Z}/p)^\times$$

is not. (For Tate's proof of the computation of $K_2(\mathbf{Q})$ see [Mil72]).

Schedule

Due to holidays and other constraints, we will lose three meetings in May. Therefore, we are considering arranging one recap session during the week of May 6th-10th and another during the week of May 27th-31st. These sessions will provide an opportunity to review material and discuss topics such as the Bott periodicity Theorem and the Atiyah-Segal completion Theorem.
Additionally, we plan to have an extra talk at the end of the semester or immediately after, focusing on the rational K-theory of number fields.

Speakers of talks 7 and 8 should work closely together.

Talk 1 - Quillen's plus-construction

(*Your name can be here!* - 18.04.2024)

- Define the algebraic K-theory space of a ring R through Quillen's $+$-construction.(References can be and [Wei13, §4] and [Rap24, §2]).
- Prove that $BGL(R)^+$ is a commutative H-group. (A reference can be [Sri96, Prop. 2.9].)
- Construct tensor product $K_i(R) \otimes K_j(R) \to K_{i+j}(R)$ (A reference can be [Sri96, Page 29].)

Talk 2 - Complex K-theory & Bott Periodicity
(*Your name can be here!* - 25.04.2024)

- Mini review of complex vector bundles and the classifying spaces for $U(n)$, U, $GL_n(k)$, $GL(k)$. (Reference can be [Kar78, §1], [May99, §23], and [Wei13, §1].)
- Introduce the (topological) complex K-theory of a CW-complex. (References can be [May99, §24] and [Wei13, §2].)
- Construct the K-theory Ω spectrum KU, and show that K is a generalizd cohomology theory. (References can be [May99, §24] and [Wei13, §2].)

Talk 3 - Adams Operations & The space $F\psi^q$
(*Your name can be here!* - 02.05.2024)

- Give a characterization of Adams operations in K-theory (References can be [May99, §24.5] and [Wei13, §2]).
- Introduce the space $F\psi^q$. (References can be [Mit, §1] and [Qui72b, §1].)
- Prove $F\psi^q$ is a simple space and computes its homotopy groups. (A reference can be [Mit, §1] .)
- Compute its rational and mod-p reduced homology as in [Mit, §1]
- Start the computation of the cohomology mod ℓ, $H^*(F\psi^q)$ as in [Qui72b, §2].

Talk 4 - The classes e_{jr}
(*Your name can be here!* - 23.05.2024)

- Explicitly construct the classes $e_{jr} \in H^{2jr-1}$ as in [Qui72b, §3].
- Recall μ and ν from [Qui72b, §1] and prove the product formula [Qui72b, Lemma 7].

Talk 5 - $H^*(F\psi^q)$ and $H_*(F\psi^q)$
(*Your name can be here!* - 06.06.2024)

- Study the algebras $H^*(F\psi^q)$ and $H_*(F\psi^q)$ from [Qui72b, §5, §6]

Talk 6 - Brauer Lifts
(*Your name can be here!* - 13.06.2024)

- Explain the Brauer lifts using [Mit, §2] and [Qui72b, §7].

Talk 7 - Homology of GL_nk part 1
(*Your name can be here!* - 20.06.2024)

- Prove that the rational and mod p homology of $BGL(k)$ vanish following [Mit, §3].
- Start discussing the mod ℓ homology of $BGL(k)$ following [Qui72b, §8]

Talk 8 - Homology of $GL_n k$ part 2
(*Your name can be here!* - 27.06.2024)

- Finish the discussion the mod ℓ homology of $BGL(k)$ following [Qui72b, §8]

Talk 9 - K-theory of finite fields
(*Your name can be here!* - 04.07.2024)

- Complete the proof of Theorem 2 following [Qui72b, §9, §12].

Talk 10 - K-th of Algebraically closed fields
(*Your name can be here!* - 11.07.2024)

- Compute the K-theory of the closure k of $\mathbf{F}_p$ following [Qui72b, §12].
- Show that this coincide with the K-theory of any algebraically closed fields k' over k following [Sus83].

Talk 11 - K-theory of Local fields
(*Your name can be here!* - 18.07.2024)

- Compute the K-theory of the closure of $\mathbf{C}$ and $\mathbf{R}$ following [Sus84].
- Discuss [Sus84] in some extent.

Extra Talk - Rational K-theory of Number Fields
(*Niklas* - Last week of semester, or the week after)

- Give a sketch of proof of Theorem 6. (A reference for this is [Bor74]. [Bes14] can be helpful too.)

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